MA117. For which natural numbers n can the set $\{1, 2, \ldots, n\}$ be partitioned into two subsets so that the sum of numbers in one subset equals to the product of the numbers in the other subset?

Originally Problem 15 from Savin Contest, Kvant 2020(11-12), proposed by V. Letzko.

We received 6 submissions, of which only 1 was correct and complete as the majority of the solvers forgot to consider the case $n = 1$. We present the solution by the Missouri State University Problem Solving Group.

Such subsets can be found for any $n \neq 2, 4$. We will denote the set we take the sum over by S and the set we take the product over by P. If $n = 1$, we take $S = \{1\}$ and $P = \{\}$ and recall that the product over an empty index set is 1. Clearly no solution is possible when $n = 2$. When $n = 3$, we take $S = \{1, 2\}$ and $P = \{3\}$. Exhaustive enumeration shows there is no solution for $n = 4$. If $n = 2k + 1$ with $k > 1$, we take

$$P = \{1, k, 2k\}$$

and S to be its complement. Then

$$\begin{aligned}\sum_{i \in S} i &= \frac{n(n+1)}{2} - 1 - k - 2k \\ &= (2k+1)(k+1) - 3k - 1 = 2k^2 = 1 \cdot k \cdot (2k) = \prod_{i \in P} i.\end{aligned}$$

If $n = 2k$ with $k > 2$, we take

$$P = \{1, k-1, 2k\}$$

and S to be its complement. Then

$$\begin{aligned}\sum_{i \in S} i &= \frac{n(n+1)}{2} - 1 - (k-1) - 2k \\ &= k(2k+1) - 3k = 2k^2 - 2k = 1 \cdot (k-1) \cdot (2k) = \prod_{i \in P} i.\end{aligned}$$

MA118. Can you colour all natural numbers using exactly 7 colours so that the product of any two (not necessarily distinct) numbers of the same colour results in a number of that same colour? For example, if 3 and 4 and coloured red, then 9, 12 and 16 must also be coloured red.

Originally Problem 5 from Savin Contest, Kvant 2020(10), proposed by M. Evdokimov.

We received 3 submissions, of which 2 were correct and complete. We present the solution by the Missouri State University Problem Solving Group.

This can be done using k colours for any positive integer k. The case $k = 1$ is trivial. For $k > 1$, let $p_1, \dots, p_{k-1}$ be distinct primes. For each $i = 1, \dots, k-1$ define the set

$$S_i = \left\{ p_i^j \middle| j \geq 1 \right\}.$$

Colour S_i with the i^{th} colour, and

$$T = \mathbb{N} \setminus \left(\cup_{i=1}^{k-1} S_i \right)$$

with the k^{th} colour. The S_i are clearly closed under multiplication. Suppose $x, y \in T$, but $xy \notin T$. Then $xy \in S_i$ for some i, which means $xy = p_i^j$ for some $j > 0$. This forces $x = p_i^\ell$ and $y = p_i^{j-\ell}$ for some $0 \leq \ell \leq j$. One of ℓ or $j - \ell$ must be non-zero, without loss of generality say $\ell \neq 0$. Then $x \in S_i$ contradicting the fact that $x \in T$. Therefore T is also closed under multiplication.

MA120. With grid paper and pencil, it is easy to draw a right-angle triangle with vertices on intersections of grid lines and with integer side-lengths; for example, the so-called Egyptian triangle with side 3, 4 and 5 will do. Can you draw a right-angle triangle with vertices on intersections of grid lines and with integer side-lengths, but so that none of its sides follows grid lines?

Originally Problem 18 from Savin Contest, Kvant 2021(1).

We received 6 submissions, all correct. We present the solution by the Missouri State University Problem Solving Group.

The answer is yes. Let (a, b, c) and (p, q, r) with $aq \neq bp$ be Pythagorean triples, that is $a^2 + b^2 = c^2, p^2 + q^2 = r^2$, and $a, b, c, p, q, r \in \mathbb{Z}^+$. The complex numbers 0, a, and bi form the vertices of a right triangle with integer side lengths. If we multiply each of these by $p + qi$, we will still have a right triangle with integer side lengths, since this multiplication corresponds to a rotation followed by a dilation by a factor of r. Performing this operation and converting to cartesian coordinates gives the vertices

$$A = (0, 0), B = (ap, aq), C = (-bq, bp).$$

Clearly neither $\overrightarrow{AB}$ nor $\overrightarrow{AC}$ is parallel to the coordinate axes. Since

$$\overrightarrow{CB} = (ap + bq, aq - bp),$$

and we chose p and q so that $aq - bp \neq 0$, this is not parallel to the coordinate axes either.

Although the argument above shows the side lengths are integers, we will also verify this directly:

$$\begin{aligned} AB &= a\sqrt{p^2 + q^2} \\ &= ar \\ AC &= b\sqrt{p^2 + q^2} \\ &= br \\ BC &= \sqrt{(ap + bq)^2 + (aq - bp)^2} \\ &= \sqrt{a^2 + b^2}\sqrt{p^2 + r^2} \\ &= cr. \end{aligned}$$

For example, taking $(a, b) = (3, 4)$ and $(p, q) = (4, 3)$, we obtain

$$A = (0, 0), B = (12, 16), C = (-12, 9)$$

giving a right triangle with sides of length 20, 15, and 25.